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Gradient Estimates and Harnack Inequalities for Nonlinear Parabolic and Nonlinear Elliptic Equations on Riemannian Manifolds

LI JIAYU

Department of Mathematics, Anhui University, Hefei, People's Republic of China

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We derive the gradient estimates and Harnack inequalities for positive solutions of nonlinear parabolic and nonlinear elliptic equations $(\Delta - \partial/\partial t)\, u(x, t) + h(x, t)\, u^\alpha(x, t) = 0$ and $\Delta u + b \cdot \nabla u + hu^\alpha = 0$ on Riemannian manifolds. We also obtain a theorem of Liouville type for positive solutions of the nonlinear elliptic equation.

0. INTRODUCTION

Let M be a complete Riemannian manifold. We consider the parabolic equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x, t) + h(x, t)\, u^\alpha(x, t) = 0 \tag{0.1}$$

on $M \times [0, \infty)$, where $h(x, t)$ is a function defined on $M \times [0, \infty)$ which is C^2 in the first variable and C^1 in the second variable, and α is a positive constant. When $\alpha = 1$, (0.1) is a linear equation; P. Li and S. T. Yau [5] derived the gradient estimates and the Harnack inequalities for positive solutions of (0.1). In this paper, we consider the general case (Theorem 2.2), and particularly we have the following global gradient estimate and Harnack inequality.

THEOREM A. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor $R_{ij} \geqslant -kg_{ij}$ $(k \geqslant 0)$. Let $h(x, t)$ be a nonnegative function defined on $M \times [0, \infty)$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that $0 < \alpha < n/(n-1)$ and $(\Delta + \partial/\partial t)\, h(x, t) \geqslant 0$.*

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If $u(x, t)$ is a positive solution of (0.1), *then*

$$\frac{|\nabla u|^2}{u^2}+\frac{1}{\alpha}hu^{\alpha-1}-\frac{1}{\alpha}\frac{u_t}{u}\leqslant\frac{2n}{\alpha^2}\frac{1}{t}+\frac{2n}{\alpha(2-\alpha)}k$$

and

$$u(x_1, t_1)\leqslant u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^{2n/\alpha}\exp\left(\frac{r^2(x_1, x_2)}{4\alpha(t_2-t_1)}+\frac{2nk}{(2-\alpha)}(t_2-t_1)\right),$$

where x_1, $x_2\in M$, $0<t_1<t_2<\infty$, and $r(x_1, x_2)$ is the geodesic distance between x_1 and x_2.

In the case where $\alpha=1$, Theorem A shows that if $h(x, t)$ is nonnegative and independent of t, the condition on the growth of $|\nabla h(x)|$ in the result of Li-Yau [5] is unnecessary.

The other purpose of this paper is to derive a gradient estimate, the Harnack inequality, and a theorem of Liouville type for positive solutions of

$$\Delta u+b\cdot\nabla u+h\cdot u^\alpha=0, \tag{0.2}$$

where $h\in C^2(M)$, $b\in\mathfrak{X}(M)$, $0<\alpha<n/(n-2)$.

B. Gidas and J. Spruck [4] proved the following theorem.

THEOREM B. *Let M be a complete Riemannian manifold of dimension n with nonnegative Ricci curvature. Suppose that $h\in C^2(M)$, $\alpha\in R^+$ satisfy the following conditions*

(1) $\forall x\in M$, $h(x)\geqslant 0$;

(2) $\forall x\in M$, $\Delta h(x)\geqslant 0$;

(3) *for $r(x)$ large $|\nabla \log h(x)|\leqslant C/r(x)$ and if $n\geqslant 4$, $h(x)\geqslant C(r(x))^\sigma$ with $\sigma>-2/(n-3)$, where $r(x)$ is the geodesic distance between x and some fixed point P*;

(4) $1\leqslant\alpha<(n+2)/(n-2)$.

If $u(x)$ is a nonnegative solution of $\Delta u+hu^\alpha=0$, then $u(x)\equiv 0$.

In this paper we show that when $1<\alpha<n/(n-2)$ $(n\geqslant 4)$, the condition (3) is unnecessary and the condition (1) can be replaced by the condition that there exists a point x_0 in M such that $h(x_0)\geqslant 0$. When $\alpha=1$ P. Li and S. T. Yau [5] proved the same result as ours under the condition that $|\nabla h(x)|=o(r(x))$ as $r(x)\to\infty$. It is surprising that when $1<\alpha<n/(n-2)$ the condition on the growth of $|\nabla h|$ is not needed.

1. Some Lemmas

In this section, we prove some lemmas which are essential to the derivation of the gradient estimate.

Suppose $u(x, t)$ is a positive solution of (0.1), that is

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x, t) = -h(x, t)\, u^{\alpha}(x, t)$$

$$u(x, t) > 0.$$

We define

$$W(x, t) = (u(x, t))^{-\beta}, \tag{1.1}$$

where β is a positive constant to be fixed,

$$\nabla W = -\beta u^{-\beta-1} \cdot \nabla u$$

$$|\nabla W|^2 = \beta^2 u^{-2(\beta+1)} |\nabla u|^2$$

$$\frac{|\nabla W|^2}{W^2} = \beta^2 \frac{|\nabla u|^2}{u^2} \tag{1.2}$$

$$W_t = \frac{\partial W}{\partial t} = -\beta u^{-\beta-1} u_t$$

$$\frac{W_t}{W} = -\beta \frac{u_t}{u}. \tag{1.3}$$

$$\begin{aligned}\Delta W &= \beta(\beta+1) u^{-(\beta+2)} |\nabla u|^2 - \beta u^{-\beta-1} \cdot \Delta u \\ &= \frac{\beta+1}{\beta} \frac{|\nabla W|^2}{W} + \beta h W^{1+(1-\alpha)/\beta} + W_t.\end{aligned}$$

Therefore,

$$\left(\Delta - \frac{\partial}{\partial t}\right) W = \frac{\beta+1}{\beta} \frac{|\nabla W|^2}{W} + \beta h W^{1+(1-\alpha)/\beta}. \tag{1.4}$$

We introduce three new functions,

$$\begin{aligned}\varphi_0(x, t) &= \frac{|\nabla W|^2}{W^2} + v h W^{(1-\alpha)/\beta} \\ \varphi_1(x, t) &= \frac{W_t}{W} \\ \varphi &= \varphi_0 + \delta \varphi_1,\end{aligned} \tag{1.5}$$

where v and δ are two constants to be fixed.

Let $e_1, e_2, \ldots, e_n$ be a local orthonormal frame field. By adopting the notation of moving frames, subscripts in i, j, and k will denote covariant differentiation in the e_i, e_j, and e_k directions where $1 \leqslant i\ j,\ k \leqslant n$.

A straightforward computation gives

$$\nabla\varphi_0(x,t)=\frac{2W_iW_{ij}}{W^2}-\frac{2W_i^2W_j}{W^3}+vW^{(1-\alpha)/\beta}h_j+v\left(\frac{1-\alpha}{\beta}\right)hW^{(1-\alpha)/\beta-1}W_j \tag{1.6}$$

$$\begin{aligned}\Delta\varphi_0(x,t)=&\frac{2W_{ij}^2}{W^2}+\frac{2W_iW_{ijj}}{W^2}-8\frac{W_iW_{ij}W_j}{W^3}+6\frac{W_i^4}{W^4}+vW^{(1-\alpha)/\beta}\cdot\Delta h\\&-2\frac{W_i^2W_{jj}}{W^3}+2v\left(\frac{1-\alpha}{\beta}\right)W^{(1-\alpha)/\beta-1}W_jh_j\\&+vh\left(\frac{1-\alpha}{\beta}\right)\left(\frac{2-\alpha}{\beta}\right)W^{(1-\alpha)/\beta}\frac{W_j^2}{W^2}\\&+vh\left(\frac{1-\alpha}{\beta}\right)W^{(1-\alpha)/\beta-1}\Delta W\end{aligned} \tag{1.7}$$

$$\frac{\partial\varphi_0(x,t)}{\partial t}=\frac{2W_iW_{it}}{W^2}-\frac{2W_i^2W_t}{W^3}+vh_tW^{(1-\alpha)/\beta}+v\left(\frac{1-\alpha}{\beta}\right)hW^{(1-\alpha)/\beta-1}W_t \tag{1.8}$$

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\varphi_1=&2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi_1\cdot\nabla\log W\\&+(1-\alpha)\,hW^{(1-\alpha)/\beta-1}W_t+\beta h_tW^{(1-\alpha)/\beta}.\end{aligned} \tag{1.9}$$

We denote the Ricci tensor of M by R_{ij}:

$$\frac{2W_iW_{ijj}}{W^2}=\frac{2W_iW_{jji}}{W^2}+\frac{2R_{ij}W_iW_j}{W^2}.$$

So,

$$\begin{aligned}\frac{2W_iW_{ijj}}{W^2}-\frac{2W_iW_{it}}{W^2}=&\frac{2W_i}{W^2}(\Delta W-W_t)_i+\frac{2R_{ij}W_iW_j}{W^2}\\=&\frac{4(\beta+1)}{\beta}\frac{W_iW_{ij}W_j}{W^3}-\frac{2(\beta+1)}{\beta}\frac{W_i^4}{W^4}+2\beta W^{(1-\alpha)/\beta-1}W_ih_i\\&+2\beta\left(1+\frac{1-\alpha}{\beta}\right)hW^{(1-\alpha)/\beta-2}W_i^2+\frac{2R_{ij}W_iW_j}{W^2}\end{aligned} \tag{1.10}$$

$$\begin{aligned}-\frac{2W_i^2W_{jj}}{W^3}+\frac{2W_i^2W_t}{W^3}=&-\frac{2W_i^2}{W^3}(\Delta W-W_t)\\=&-\frac{2(\beta+1)}{\beta}\frac{W_i^4}{W^4}-2\beta hW^{(1-\alpha)/\beta-2}W_i^2.\end{aligned} \tag{1.11}$$

By the Hölder's inequality, we have

$$\frac{2W_{ij}^2}{W^2}-8\frac{W_iW_{ij}W_j}{W^3}+6\frac{W_i^4}{W^4}\geqslant\frac{2(1-\varepsilon)W_{ij}^2}{W^2}-4\frac{W_iW_{ij}W_j}{W^3}+\left(6-\frac{2}{\varepsilon}\right)\frac{W_i^4}{W^4},$$

where $0<\varepsilon<1$.

By the inequality $W_{ij}^2\geqslant(1/n)(W_{ii})^2$, we obtain

$$\frac{2W_{ij}^2}{W^2}-8\frac{W_iW_{ij}W_j}{W^3}+6\frac{W_i^4}{W^4}$$
$$\geqslant\frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2-4\left(\frac{W_iW_{ij}W_j}{W^3}-\frac{W_i^4}{W^4}\right)-2\left(\frac{1}{\varepsilon}-1\right)\frac{W_i^4}{W^4}. \quad (1.12)$$

By (1.6),

$$\nabla\varphi_0\nabla\log W=\frac{2W_iW_{ij}W_j}{W^3}-\frac{2W_i^4}{W^4}+\nu W^{(1-\alpha)/\beta-1}h_iW_i$$
$$+\nu\left(\frac{1-\alpha}{\beta}\right)hW^{(1-\alpha)/\beta-2}W_i^2. \quad (1.13)$$

Substituting (1.10), (1.11), (1.12), and (1.13) into (1.7) and (1.8) we obtain

$$\left(\Delta-\frac{\partial}{\partial t}\right)\varphi_0\geqslant\frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\frac{W_i^4}{W^4}+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi_0\cdot\nabla\log W$$
$$+\left[2(1-\alpha)+\nu\frac{\alpha(\alpha-1)}{\beta^2}\right]hW^{(1-\alpha)/\beta}\frac{W_i^2}{W^2}+\nu(1-\alpha)h^2W^{2(1-\alpha)/\beta}$$
$$+2\beta\left(1-\frac{\alpha}{\beta^2}\nu\right)W^{(1-\alpha)/\beta-1}W_ih_i+\nu W^{(1-\alpha)/\beta}\Delta h$$
$$+\frac{2R_{ij}W_iW_j}{W^2}-\nu W^{(1-\alpha)/\beta}h_t. \quad (1.14)$$

We set $\delta=\nu/\beta$; then

$$\left(\Delta-\frac{\partial}{\partial t}\right)\varphi\geqslant\frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\frac{W_i^4}{W^4}+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi\cdot\nabla\log W$$
$$+(1-\alpha)\left(2-\nu\frac{\alpha}{\beta^2}\right)hW^{(1-\alpha)/\beta}\frac{W_j^2}{W^2}+\nu(1-\alpha)h^2W^{2(1-\alpha)/\beta}$$
$$+2\beta\left(1-\nu\frac{\alpha}{\beta^2}\right)W^{(1-\alpha)/\beta-1}W_ih_i+\frac{\nu(1-\alpha)}{\beta}hW^{(1-\alpha)/\beta-1}W_t$$
$$+\nu W^{(1-\alpha)/\beta}\cdot\Delta h+\frac{2R_{ij}W_iW_j}{W^2}. \quad (1.15)$$

By (1.4),

$$\frac{\Delta W}{W}=\frac{\beta+1}{\beta}\frac{|\nabla W|^2}{W^2}+\beta h W^{(1-\alpha)/\beta}+\frac{W_t}{W}$$

$$=\frac{\beta+1}{\beta}\frac{|\nabla W|^2}{W^2}+\frac{\beta}{\nu}\left(\varphi-\frac{|\nabla W|^2}{W^2}\right)$$

$$=\frac{\beta}{\nu}\varphi+\left(\frac{\beta+1}{\beta}-\frac{\beta}{\nu}\right)\frac{|\nabla W|^2}{W^2}. \tag{1.16}$$

If we set $\nu=s(\beta^2/\alpha)$, then (1.16) becomes

$$\frac{\Delta W}{W}=\frac{1}{s}\frac{\alpha}{\beta}\varphi+\left(\frac{\beta+1-\alpha/s}{\beta}\right)\frac{|\nabla W|^2}{W^2}. \tag{1.17}$$

Substituting (1.17) into (1.15), we obtain

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\varphi \geqslant{}& \frac{2(1-\varepsilon)}{n}\frac{1}{s^2}\frac{\alpha^2}{\beta^2}\varphi^2\\
&+\left[\frac{2(1-\varepsilon)}{n}\frac{(\alpha-s\beta-s)^2}{s^2\beta^2}-2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4}\\
&+\frac{4(1-\varepsilon)}{n}\frac{\alpha(s\beta+s-\alpha)}{s^2\beta^2}\varphi\frac{|\nabla W|^2}{W^2}+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi\cdot\nabla\log W\\
&+(1-\alpha)(2-s)\,hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2}+\nu(1-\alpha)\,h^2W^{2(1-\alpha)/\beta}\\
&+\frac{\nu}{\beta}(1-\alpha)\,hW^{(1-\alpha)/\beta}\frac{W_t}{W}\\
&+2\beta(1-s)\,W^{(1-\alpha)/\beta-1}\,W_ih_i+\nu W^{(1-\alpha)/\beta}\,\Delta h\\
&+\frac{2W_iR_{ij}W_j}{W^2}.\end{aligned} \tag{1.18}$$

Equation (1.18) implies the following lemma.

LEMMA 1.1. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. If φ is defined by* (1.5) *where $\delta=\nu/\beta$, $\nu=s(\beta^2/\alpha)$, then*

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\varphi \geqslant & \frac{2(1-\varepsilon)}{n}\frac{1}{s^2}\frac{\alpha^2}{\beta^2}\varphi^2\\
&+\left[\frac{2(1-\varepsilon)}{n}\frac{(\alpha-s\beta-s)^2}{s^2\cdot\beta^2}-2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4}\\
&+\frac{4(1-\varepsilon)}{n}\frac{\alpha(s\beta+s-\alpha)}{s^2\cdot\beta^2}\varphi\frac{|\nabla W|^2}{W^2}+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi\cdot\nabla\log W\\
&+(1-\alpha)\,hW^{(1-\alpha)/\beta}\varphi+(\alpha-1)(s-1)\,hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2}\\
&+2\beta(1-s)\,W^{(1-\alpha)/\beta-1}W_ih_i\\
&+vW^{(1-\alpha)/\beta}\cdot\Delta h+\frac{2W_iR_{ij}W_j}{W^2}.\end{aligned}\tag{1.19}$$

If we set $\delta=2(v/\beta)$, *then* (1.14) *yields*

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\varphi \geqslant & \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\frac{|\nabla W|^4}{W^4}\\
&+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi\cdot\nabla\log W\\
&+(1-\alpha)\left(2-v\frac{\alpha}{\beta^2}\right)hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2}+v(1-\alpha)\,h^2W^{2(1-\alpha)/\beta}\\
&+2\beta\left(1-v\frac{\alpha}{\beta^2}\right)W^{(1-\alpha)\beta-1}W_ih_i+\frac{2v}{\beta}(1-\alpha)\,hW^{(1-\alpha)\beta-1}W_t\\
&+vW^{(1-\alpha)/\beta}\left(\Delta h+\frac{\partial h}{\partial t}\right)+\frac{2R_{ij}W_iW_j}{W^2}.\end{aligned}\tag{1.20}$$

By (1.4),

$$\frac{\Delta W}{W}=\left(\frac{\beta+1-\alpha/2}{\beta}\right)\frac{|\nabla W|^2}{W^2}+\frac{\alpha}{2\beta}\varphi+\frac{\beta}{2}hW^{(1-\alpha)/\beta},$$

where $v=\beta^2/\alpha$. *So,*

$$\begin{aligned}\left(\frac{\Delta W}{W}\right)^2=&\left(\frac{\beta+1-\alpha/2}{\beta}\right)^2\frac{|\nabla W|^4}{W^4}+\frac{\alpha}{\beta}\left(\frac{\beta+1-\alpha/2}{\beta}\right)\varphi\frac{|\nabla W|^2}{W^2}\\
&+\left(\beta+1-\frac{\alpha}{2}\right)\frac{|\nabla W|^2}{W^2}hW^{(1-\alpha)/\beta}\\
&+\frac{\alpha^2}{4\beta^2}\varphi^2+\frac{\alpha}{2}\varphi hW^{(1-\alpha)/\beta}+\frac{\beta^2}{4}h^2W^{2(1-\alpha)/\beta}.\end{aligned}$$

Substituting this identity into (1.20) and setting $\nu=\beta^2/\alpha$, we note that (1.20) yields the following lemma:

LEMMA 1.2. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. Let $h(x,t)$ be a function defined on $M\times[0,\infty)$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that $h(x,t)\geqslant 0$, $(\Delta+\partial/\partial t)\,h(x,t)\geqslant 0$, and $0<\alpha<n/(n-1)$.*

We define

$$\varphi=\frac{|\nabla W|^2}{W^2}+\frac{\beta^2}{\alpha}hW^{(1-\alpha)/\beta}+\frac{2\beta}{\alpha}\frac{W_t}{W};$$

if $\varphi(x_0)>0$, then at $x_0\in M$ we have

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)\varphi\geqslant{}&\frac{2(1-\varepsilon)}{n}\frac{\alpha^2}{4\beta^2}\varphi^2+\frac{2(1-\varepsilon)}{n}\frac{\alpha(\beta+1-\alpha/2)}{\beta^2}\varphi\frac{|\nabla W|^2}{W^2}\\&+\left[\frac{2(1-\varepsilon)}{n}\left(1+\frac{1-\alpha/2}{\beta}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4}\\&+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\varphi\cdot\nabla\log W+\frac{2W_iR_{ij}W_j}{W^2},\end{aligned}$$

where β is a positive constant to be fixed and ε is a positive constant such that $0<\varepsilon<n(1/n-(\alpha-1)/\alpha)$.

We now consider properties of positive solutions of (0.2). Suppose $u(x)$ is a positive solution of (0.2) and set

$$W(x)=u^{-\beta}(x),$$

where β is also a positive constant to be fixed:

$$\nabla W=-\beta u^{-\beta-1}\nabla u \tag{1.21}$$

$$\frac{|\nabla W|^2}{W^2}=\beta^2\frac{|\nabla u|^2}{u^2} \tag{1.22}$$

$$\begin{aligned}\Delta W&=\beta(\beta+1)\,u^{-\beta-2}|\nabla u|^2-\beta u^{-\beta-1}\Delta u\\&=\frac{\beta+1}{\beta}\frac{|\nabla W|^2}{W}+\beta hW^{1+(1-\alpha)/\beta}-b\cdot\nabla W.\end{aligned} \tag{1.23}$$

We introduce another function

$$\psi(x)=\frac{|\nabla W|^2}{W^2}+\nu hW^{(1-\alpha)/\beta}. \tag{1.24}$$

Assume $b=b_i\cdot e_i$. Using (1.23), we get, after a straightforward computation,

$$\Delta\psi=\frac{2W_{ij}^2}{W^2}+\frac{2W_iW_{ijj}}{W^2}-8\frac{W_iW_{ij}W_j}{W_3}-2\frac{W_i^2W_{jj}}{W^3}+6\frac{W_i^4}{W^4}+vW^{(1-\alpha)/\beta}h_{jj}$$

$$+2v\left(\frac{1-\alpha}{\beta}\right)W^{(1-\alpha)/\beta-1}W_jh_j+vh\left(\frac{1-\alpha}{\beta}\right)\left(\frac{2-\alpha}{\beta}\right)W^{(1-\alpha)/\beta}\frac{W_j^2}{W^2}$$

$$+v(1-\alpha)\,h^2(x)\,W^{2(1-\alpha)/\beta}+v\left(\frac{\alpha-1}{\beta}\right)hW^{(1-\alpha)/\beta-1}b_i\cdot W_i.$$

The following estimate can be obtained in a similar manner as Lemma 1.1:

$$\Delta\psi\geqslant\frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\frac{|\nabla W|^4}{W^4}+2\left(\frac{\beta+1}{\beta}-1\right)\nabla\psi\cdot\nabla\log W$$

$$+\left[2(1-\alpha)+v\frac{\alpha(\alpha-1)}{\beta^2}\right]hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2}+v(1-\alpha)\,h^2W^{2(1-\alpha)/\beta}$$

$$+2\beta\left(1-v\frac{\alpha}{\beta^2}\right)W^{(1-\alpha)/\beta-1}W_ih_i+vW^{(1-\alpha)/\beta}(\Delta h+\nabla h\cdot b)$$

$$+\frac{2W_iR_{ij}W_j}{W^2}-2\frac{W_ib_{ij}W_j}{W^2}-\nabla\psi\cdot b. \tag{1.25}$$

By (1.23), we have

$$\frac{\Delta W}{W}=\frac{\beta+1}{\beta}\frac{|\nabla W|^2}{W^2}+\beta hW^{(1-\alpha)/\beta}-b\frac{\nabla W}{W}$$

$$=\left(\frac{\beta+1}{\beta}-\frac{\beta}{v}\right)\frac{|\nabla W|^2}{W^2}-\frac{\beta}{v}\psi-b\cdot\frac{\nabla W}{W}$$

$$\left(\frac{\Delta W}{W}\right)^2=\left(\frac{\beta}{v}\right)^2\psi^2+\left(\frac{\beta+1}{\beta}-\frac{\beta}{v}\right)^2\frac{|\nabla W|^4}{W^4}+|b|^2\cdot\frac{|\nabla W|^2}{W^2}$$

$$+2\frac{\beta}{v}\left(\frac{\beta+1}{\beta}-\frac{\beta}{v}\right)\psi\frac{|\nabla W|^2}{W^2}$$

$$-\frac{2\beta}{v}\psi b\frac{\nabla W}{W}-2\left(\frac{\beta+1}{\beta}-\frac{\beta}{v}\right)b\frac{\nabla W}{W}\frac{|\nabla W|^2}{W^2}. \tag{1.26}$$

By Hölder's inequality, we have

$$\frac{2\beta}{\nu}\psi b\frac{\nabla W}{W} \leqslant \varepsilon'\left(\frac{\beta}{\nu}\right)^2\psi^2 + \frac{1}{\varepsilon'}|b|^2\frac{|\nabla W|^2}{W^2} \tag{1.27}$$

$$2\left(\frac{\beta+1}{\beta}-\frac{\beta}{\nu}\right)b\frac{\nabla W}{W}\frac{|\nabla W|^2}{W^2} \leqslant \varepsilon''\left(\frac{\beta+1}{\beta}-\frac{\beta}{\nu}\right)^2\frac{|\nabla W|^4}{W^4} + \frac{1}{\varepsilon''}|b|^2\frac{|\nabla W|^2}{W^2}, \tag{1.28}$$

where $\varepsilon', \varepsilon'' > 0$.

Substituting (1.26), (1.27), and (1.28) into (1.25), we have

$$\begin{aligned} \Delta\psi \geqslant{}& \frac{2(1-\varepsilon)}{n}(1-\varepsilon')\left(\frac{\beta}{\nu}\right)^2\psi^2 + \frac{2(1-\varepsilon)}{n}(1-\varepsilon'')\left(\frac{\beta+1}{\beta}-\frac{\beta}{\nu}\right)^2\frac{|\nabla W|^4}{W^4} \\ &+ \frac{4(1-\varepsilon)}{n}\frac{\beta}{\nu}\left(\frac{\beta+1}{\beta}-\frac{\beta}{\nu}\right)\psi\frac{|\nabla W|^2}{W^2} - 2\left(\frac{1}{\varepsilon}-1\right)\frac{|\nabla W|^4}{W^4} \\ &+ 2\left(\frac{\beta+1}{\beta}-1\right)\nabla\psi\cdot\nabla\log W \\ &+ \left[2(1-\alpha)+\nu\frac{\alpha(\alpha-1)}{\beta^2}\right]hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2} + \nu(1-\alpha)h^2W^{2(1-\alpha)/\beta} \\ &+ 2\beta\left(1-\frac{\alpha}{\beta^2}\nu\right)W^{(1-\alpha)/\beta-1}W_ih_i + \nu W^{(1-\alpha)/\beta}(\Delta h + \nabla h\cdot b) - \nabla\psi\cdot b \\ &+ \frac{2W_i(R_{ij}-b_{ij})W_j}{W^2} - \frac{2(1-\varepsilon)}{n}\left(\frac{1}{\varepsilon'}+\frac{1}{\varepsilon''}-1\right)|b|^2\frac{|\nabla W|^2}{W^2}. \end{aligned} \tag{1.29}$$

We set $\nu = \beta^2/\alpha$; then

$$\begin{aligned} \nu(1-\alpha)h^2W^{2(1-\alpha)/\beta} &+ \left[2(1-\alpha)+\nu\frac{\alpha}{\beta^2}(\alpha-1)\right]hW^{(1-\alpha)/\beta}\frac{|\nabla W|^2}{W^2} \\ &= (1-\alpha)hW^{(1-\alpha)/\beta}\psi \\ &= \frac{\alpha^2}{\beta^2}\frac{1-\alpha}{\alpha}\psi^2 + \frac{\alpha(\alpha-1)}{\beta^2}\frac{|\nabla W|^2}{W^2}\psi. \end{aligned} \tag{1.30}$$

Substituting (1.30) into (1.29) yields the following lemma.

LEMMA 1.3. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. If ψ is defined by* (1.24), *then*

$$\begin{aligned}
\Delta\psi \geqslant & \left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon')-\frac{\alpha-1}{\alpha}\right]\frac{\alpha^2}{\beta^2}\psi^2 \\
& +\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon'')\left(\frac{\beta+1-\alpha}{\beta}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4} \\
& +\left[\frac{4(1-\varepsilon)}{n}\frac{\alpha(\beta+1-\alpha)}{\beta^2}+\frac{\alpha(\alpha-1)}{\beta^2}\right]\frac{|\nabla W|^2}{W^2}\psi \\
& +2\left(\frac{\beta+1}{\beta}-1\right)\nabla\psi\cdot\nabla\log W \\
& +vW^{(1-\alpha)/\beta}(\Delta h+\nabla h\cdot b)-\nabla\psi\cdot b+\frac{2W_i(R_{ij}-b_{ij})W_j}{W^2} \\
& -\frac{2(1-\varepsilon)}{n}\left(\frac{1}{\varepsilon'}+\frac{1}{\varepsilon''}-1\right)|b|^2\frac{|\nabla W|^2}{W^2}.
\end{aligned} \tag{1.31}$$

2. Gradient Estimates and Harnack Inequalities for Parabolic Equations

In this section, we first give interior estimates and then we extend our local estimates to global ones. Finally, we prove the Harnack inequality for positive solutions of (0.1).

THEOREM 2.1. *Let M be a complete Riemannian manifold with possibly empty boundary ∂M. Assume $P \in M$, and $B_P(2R)$, the geodesic ball of radius $2R$ around P does not intersect the boundary ∂M. We denote $-K(2R)$ to be the constant such that the Ricci curvature of M is bounded from below by $-K(2R)$ in $B_P(2R)$, and $K(2R) \geqslant 0$. Let $h(x, t)$ be a function defined on $M \times [0, \infty)$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that*

$$\Delta h \geqslant -\theta(2R)$$

and for $\alpha \geqslant 1$, assume in addition that

$$|\nabla h(x, t)| \leqslant v(2R)$$

on $B_P(2R) \times [0, \infty)$ for some constants $\theta(2R)$ and $v(2R)$.

If $u(x, t)$ is a positive solution of (0.1) on $M \times [0, \infty)$, then for $0 < \alpha < 1$, $u(x, t)$ satisfies the estimate

$$\frac{|\nabla u|^2}{u^2}+\frac{1}{\alpha}hu^{\alpha-1}-\frac{1}{\alpha}\frac{u_t}{u}$$

$$\leqslant\frac{n}{2\alpha^2}\frac{1}{t}+\frac{n}{2\alpha^2}(1-\alpha)M_1M_2+\frac{\sqrt{n}}{\sqrt{2}}\frac{1}{\alpha^{3/2}}(\theta(2R)M_2)^{1/2}$$

$$+\frac{n}{2\alpha(1-\alpha)}K(2R)$$

$$+\frac{n}{2\alpha^2}\frac{1}{R^2}\left[\frac{n}{2(1-\alpha)\alpha}C_1+C_2C_3\sqrt{n-1}\,R(K(2R))^{1/2}\right], \quad (2.1)$$

where C_1, C_2, C_3 *are positive constants and*

$$M_1=\max\{h_-(x,t)\,|\,(x,t)\in B_P(2R)\times[0,\infty)\},$$
$$M_2=\max\{u^{\alpha-1}(x,t)\,|\,(x,t)\in B_P(2R)\times[0,\infty)\},$$

and for $\alpha\geqslant 1$, $u(x,t)$ *satisfies the estimate*

$$\frac{|\nabla u|^2}{u^2}+\frac{s}{\alpha}hu^{\alpha-1}-\frac{s}{\alpha}\frac{u_t}{u}$$

$$\leqslant\frac{n}{2(1-\varepsilon)}\left(\frac{s}{\alpha}\right)^2\frac{1}{t}+\frac{n}{2(1-\varepsilon)}\left(\frac{s}{\alpha}\right)^2(\alpha-1)\,M_4M_5$$

$$+\frac{\sqrt{n}}{\sqrt{2(1-\varepsilon)}}\frac{s}{\alpha}\left(M_6+\frac{s}{\alpha}\theta(2R)M_4\right)^{1/2}$$

$$+\frac{n}{2(1-\varepsilon)}\left(\frac{s}{\alpha}\right)^2\frac{1}{R^2}\left[\frac{n}{4(1-\varepsilon)}\frac{s^2C_1}{\alpha(s-\alpha)}+C_2+C_3\sqrt{n-1}\,R\sqrt{K(2R)}\right], \quad (2.2)$$

where

$$-M_6=\frac{1}{\beta}\min_{y\geqslant 0}\left\{\left[\frac{2(1-\varepsilon)}{n}\left(1+\frac{s-\alpha}{s\beta}\right)^2-2\left(\frac{1}{\varepsilon}-1\right)\right]y^2\right.$$
$$\left.-((\alpha-1)(s-1)\,M_3M_4+2K(2R)\,y-2\beta(s-1)\,M_4\nu(2R)\,y^{1/2}\right\},$$

where β *is a positive constant such that* $(1/n)(1+(s-\alpha)/s\beta)^2>1/\varepsilon$; s *is a positive constant such that* $s>\alpha$; C_1, C_2, C_3, C_4 *are positive constants*; *and*

$$M_3=\max\{h_-(x,t)\,|\,(x,t)\in B_P(2R)\times[0,\infty)\}$$
$$M_4=\max\{u^{\alpha-1}(x,t)\,|\,(x,t)\in B_P(2R)\times[0,\infty)\}$$
$$M_5=\max\{h(x,t)\,|\,(x,t)\in B_P(2R)\times[0,\infty)\}.$$

Proof. We define the function $F(x, t) = t \cdot \varphi(x, t)$. Let $\tilde{g}(r)$ be a C^2 function defined on $[0, \infty)$ such that

$$\tilde{g}(r) = \begin{cases} 1 & \text{if } r \in [0, 1] \\ 0 & \text{if } r \geqslant 2 \end{cases}$$

$0 \leqslant \tilde{g}(r) \leqslant 1$, $\tilde{g}'(r) \leqslant 0$, $\tilde{g}''(r) \geqslant -C_2$, and $|\tilde{g}'(r)|^2/\tilde{g}(r) \leqslant C_1$, where C_1, C_2 are positive constants.

If $r(x)$ denotes the geodesic distance between x and some fixed point P, we set

$$g(x) = \tilde{g}\left(\frac{r(x)}{R}\right).$$

Using the argument of Calabi [1] (also see [2]), we assume that the function $g(x) \cdot F(x, t)$ with support in $B_P(2R)$ is smooth. Let (x_0, t_0) be the point where $g \cdot F$ achieves its maximum in $B_P(2R) \times [0, T]$. Clearly, we may assume that $g(x_0) \cdot F(x_0, t_0) > 0$. By the maximum principle, at (x_0, t_0), we have

$$\nabla(g \cdot F) = 0 \tag{2.3}$$

$$\frac{\partial(g \cdot F)}{\partial t} \geqslant 0 \tag{2.4}$$

$$\Delta(g \cdot F) \leqslant 0. \tag{2.5}$$

Obviously,

$$\frac{|\nabla g|^2}{g} = \frac{1}{R^2} \frac{|\tilde{g}'|^2}{\tilde{g}} \leqslant \frac{C_1}{R^2} \tag{2.6}$$

$$\Delta g = \tilde{g}'' \frac{1}{R^2} + \frac{\tilde{g}'}{R} \cdot \Delta r.$$

Applying the Laplacian comparison theorem, we have

$$\Delta r \leqslant \frac{n-1}{r} (1 + \sqrt{K} r)$$

$$\begin{aligned} \Delta g &\geqslant -\frac{C_2}{R^2} - \frac{C_1(n-1)}{R^2} (1 + \sqrt{K(2R)} \cdot R) \\ &\geqslant -\frac{C_2 + (n-1)C_1}{R^2} - \frac{C_1(n-1)}{R} \sqrt{K(2R)}. \end{aligned} \tag{2.7}$$

By (2.3), we have

$$\nabla F = -\frac{\nabla g}{g} \cdot F. \tag{2.8}$$

By (2.5), we have

$$\Delta g \cdot F + 2 \cdot \nabla g \cdot \nabla F + g \cdot \Delta F \leqslant 0. \tag{2.9}$$

Using (2.4) and (2.9), we obtain

$$\Delta g \cdot F + 2\nabla g \cdot \nabla F + g \left(\Delta - \frac{\partial}{\partial t} \right) F \leqslant 0. \tag{2.10}$$

If $0 < \alpha < 1$, by setting $s = 1$, and using Lemma 1.1, we have

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t} \right) F \geqslant & \frac{2(1-\varepsilon)}{n} \frac{\alpha^2}{\beta^2} \frac{1}{t} F^2 + t \left[\frac{2(1-\varepsilon)}{n} \left(\frac{\beta + 1 - \alpha}{\beta} \right)^2 - 2 \left(\frac{1}{\varepsilon} - 1 \right) \right] \frac{|\nabla W|^4}{W^4} \\
& + \frac{4(1-\varepsilon)}{n} \frac{\alpha(\beta + 1 - \alpha)}{\beta^2} F \cdot \frac{|\nabla W|^2}{W^2} \\
& + \frac{2}{\beta} \nabla F \cdot \nabla \log W + (1-\alpha) h W^{(1-\alpha)/\beta} \cdot F \\
& - \frac{\beta^2}{\alpha} t \cdot \theta(2R) W^{(1-\alpha)/\beta} - 2tK(2R) \frac{|\nabla W|^2}{W^2} - \frac{F}{t}.
\end{aligned} \tag{2.11}$$

By Hölder's inequality,

$$\begin{aligned}
2t \cdot K(2R) \frac{|\nabla W|^2}{W^2} \leqslant & \frac{2(1-\varepsilon)}{n} \left(\frac{1-\alpha}{\beta} \right)^2 t \frac{|\nabla W|^4}{W^4} \\
& + \frac{n\beta^2}{2(1-\varepsilon)(1-\alpha)^2} \cdot t \cdot K^2(2R).
\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.11), if we choose $\beta > 0$ such that $1/\beta > n/2\varepsilon(1-\alpha)$, then we have

$$\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t} \right) F \geqslant & \frac{2(1-\varepsilon)}{n} \frac{\alpha^2}{\beta^2} \frac{1}{t} F^2 + \frac{4(1-\varepsilon)}{n} \frac{\alpha(\beta + 1 - \alpha)}{\beta^2} F \frac{|\nabla W|^2}{W^2} \\
& + \frac{2}{\beta} \nabla F \cdot \nabla \log W + (1-\alpha) h W^{(1-\alpha)/\beta} F \\
& - \frac{\beta^2}{\alpha} \theta(2R)\, t W^{(1-\alpha)/\beta} - \frac{n\beta^2}{2(1-\varepsilon)(1-\alpha)^2} t K^2(2R).
\end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.10) and using (2.8), we have

$$\frac{2(1-\varepsilon)}{n}\frac{\alpha^2}{\beta^2}\frac{1}{t}\, g\cdot F^2+\frac{4(1-\varepsilon)}{n}\frac{\alpha(\beta+1-\alpha)}{\beta^2}\cdot g\cdot F\frac{|\nabla W|^2}{W^2}-\frac{2}{\beta}F\cdot\nabla g\cdot\frac{\nabla W}{W}$$
$$-(1-\alpha)\,M_1M_2\,gF-\frac{\beta^2}{\alpha}\,\theta(2R)M_2t-\frac{n\beta^2}{2(1-\varepsilon)(1-\alpha)^2}\,tK^2(2R)$$
$$-\frac{gF}{t}-\left(\frac{C_2'}{R^2}-\frac{C_1(n-1)}{R}\sqrt{K(2R)}\right)gF\leqslant 0. \tag{2.14}$$

Clearly,

$$\frac{2}{\beta}\,gF\cdot\nabla g\,\frac{\nabla W}{W}\leqslant\frac{4(1-\varepsilon)}{n}\frac{\alpha(1-\alpha)}{\beta^2}\,gF\frac{|\nabla W|^2}{W^2}$$
$$+\frac{n}{4(1-\varepsilon)\,\alpha(1-\alpha)}\frac{|\nabla g|^2}{g}. \tag{2.15}$$

Multiplying through by g at (2.14) and using (2.15), we obtain

$$\frac{2(1-\varepsilon)}{n}\frac{\alpha^2}{\beta^2}\,g^2F^2-(1-\alpha)\,M_1M_2tgF-gF$$
$$-t\left(\frac{C_3}{R^2}+\frac{C_1(n-1)}{R}\sqrt{K(2R)}+\frac{C_1}{B^2}\frac{n}{4(1-\varepsilon)\,\alpha(1-\alpha)}\right)gF$$
$$-t^2\left(\frac{\beta^2}{\alpha}\,\theta(2R)M_2+\frac{n\beta^2}{2(1-\varepsilon)(1-\alpha)^2}\,K^2(2R)\right)\leqslant 0. \tag{2.16}$$

Equation (2.16) implies

$$\frac{|\nabla u|^2}{u^2}+\frac{1}{\alpha}hu^{\alpha-1}-\frac{1}{\alpha}\frac{u_t}{u}\leqslant\frac{n}{2(1-\varepsilon)}\frac{1}{\alpha^2}\frac{1}{t}+\frac{n}{2(1-\varepsilon)}\frac{(1-\alpha)}{\alpha^2}M_1M_2$$
$$+\sqrt{\frac{n}{2(1-\varepsilon)}}\left(\frac{1}{\alpha}\right)^{3/2}\sqrt{\theta(2R)M_2}+\frac{nK(2R)}{2(1-\varepsilon)\,\alpha(1-\alpha)}$$
$$+\frac{n}{2(1-\varepsilon)}\frac{1}{\alpha^2}\left[\frac{n}{2(1-\varepsilon)}\frac{1}{\alpha(1-\alpha)}\,C_1+C_2+C_3\sqrt{n-1}\cdot R\cdot\sqrt{K(2R)}\right]\frac{1}{R^2}.$$

Letting $\varepsilon\to 0$, we obtain (2.1).

If $\alpha\geqslant 1$, applying Lemma 1.1, we have

$$\left(\Delta-\frac{\partial}{\partial t}\right)F\geqslant\frac{2(1-\varepsilon)}{n}\frac{\alpha^2}{\beta^2s^2}\varphi^2+\left[\frac{2(1-\varepsilon)}{n}\left(1+\frac{s-\alpha}{\beta s}\right)^2-2\right]\frac{|\nabla W|^4}{W^4}$$
$$+\frac{4(1-\varepsilon)}{n}\frac{\alpha(s\beta+s-\alpha)}{s^2\beta^2}F\frac{|\nabla W|^2}{W^2}$$
$$+2\frac{1}{\beta}\nabla F\cdot\nabla\log W-(\alpha-1)M_4M_5F$$
$$-2s\frac{\beta^2}{\alpha}t\theta(2R)M_4-(2K(2R)+(\alpha-1)(s-1)M_3M_4)\frac{|\nabla W|^2}{W^2}$$
$$-2\beta(s-1)M_4\cdot v(2R)\frac{|\nabla W|}{W}. \tag{2.17}$$

To obtain estimate (2.2), instead of using the inequality (2.15) we can use the inequality

$$\frac{2}{\beta}gF\cdot\nabla g\frac{\nabla W}{W}\leqslant\frac{4(1-\varepsilon)}{n}\frac{\alpha(s-\alpha)}{\beta^2s^2}F\frac{|\nabla W|^2}{W^2}+\frac{n}{4(1-\varepsilon)}\frac{s^2}{\alpha(s-\alpha)}\frac{|\nabla g|^2}{g} \tag{2.18}$$

and get (2.2) by an argument similar to the one used in obtaining (2.1).

Remark. In the proof of Theorem 2.2, we show that

$$M_6\leqslant\frac{n}{4}\left(\frac{s}{s-\alpha}\right)^2((\alpha-1)(s-1)M_3M_4+2K(2R))^2$$
$$+C_4\left(\frac{n(s-1)^4s^2M^4v^4(2R)}{(s-\alpha)^2}\right)^{1/3}.$$

Now, we prove the following global estimates.

THEOREM 2.2. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}.*

Let $h(x,t)$ be a function defined on $M\times[0,\infty)$ which is C^2 in the x-variable and C^1 in the t-variable.

(I) *For $0<\alpha<1$, assume that $h\geqslant 0$, $R_{ij}\geqslant -k_1g_{ij}$ $(k_1\geqslant 0)$, and $\Delta h\geqslant 0$. If $u(x,t)$ is a positive solution of (0.1) on $M\times[0,\infty)$, then*

$$\frac{|\nabla u|^2}{u^2}+\frac{1}{\alpha}hu^{\alpha-1}-\frac{1}{\alpha}\frac{u_t}{u}\leqslant\frac{n}{2\alpha^2}\frac{1}{t}+\frac{n}{\alpha(1-\alpha)}k_1. \tag{2.19}$$

(II) *For $\alpha=1$, assume that $R_{ij}\geqslant 0$, $\Delta h\geqslant -k_2$ $(k_2\geqslant 0)$, and $|\nabla h|=o(r(x))$ as $r(x)\to\infty$, where $r(x)$ is the geodesic distance between x and some* *fixed point* $P \in M$. *If* $u(x, t)$ *is a positive solution of* (0.1) *on* $M \times [0, \infty)$, *then*

$$\frac{|\nabla u|^2}{u^2} + h - \frac{u_t}{u} \leqslant \frac{n}{2}\frac{1}{t} + \sqrt{\frac{n}{2}}\,k_2. \tag{2.20}$$

(III) *For* $\alpha \geqslant 1$, *assume that* $R_{ij} \geqslant -k_1 g_{ij}$, $\Delta h \geqslant -k_2$, $-k_3 \leqslant h \leqslant k_4$, *and* $|\nabla h| \leqslant k_5$ $(k_1, k_2, k_3, k_4, k_5 \geqslant 0)$. *If* $u(x, t)$ *is a bounded positive solution of* (0.1), *then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} + \frac{s}{\alpha} h \cdot u^{\alpha - 1} - \frac{s}{\alpha}\frac{u_t}{u} \\
&\quad \leqslant \frac{n}{2}\left(\frac{s}{\alpha}\right)^2 \frac{1}{t} + \frac{n}{2}\frac{s^2}{\alpha(s-\alpha)} k_1 + \frac{n}{2}\left(\frac{s}{\alpha}\right)^2 (\alpha - 1) k_4 M_0 \\
&\qquad + \frac{n}{2\sqrt{2}}\frac{s^2}{\alpha(s-\alpha)}(\alpha-1)(s-1)k_3 M_0 \\
&\qquad + C_4\sqrt{n}\,\frac{s}{\alpha}\left(\frac{\sqrt{n}\,s(s-1)^2 k_5^2}{s-\alpha}\right)^{1/3} M_0^{2/3} + \sqrt{\frac{n}{2}}\left(\frac{s}{\alpha}\right)^{3/2}\sqrt{k_2}\,M_0^{1/2},
\end{aligned} \tag{2.21}$$

where $M_0 = \operatorname{Sup}\{u^{\alpha-1}(x, t) \mid (x, t) \in M \times [0, \infty)\}$, s *is a positive constant such that* $s > \alpha$, *and* $C_4 > 0$ *is an absolute constant.*

Proof. By setting $\theta(2R) = M_1 = 0$ and $K(2R) = k_1$, and letting $R \to \infty$, we note that (2.1) yields (2.19).

In order to prove (2.20) and (2.21), we compute M_6. We set

$$A = (\alpha - 1)(s-1)\, M_3 M_4 + 2K(2R).$$

Clearly,

$$A \cdot y \leqslant \frac{n}{4(1-\varepsilon)}\left(\frac{s\beta}{s-\alpha}\right)^2 A^2 + \frac{1-\varepsilon}{n}\left(\frac{s-\alpha}{s\beta}\right)^2 y^2.$$

Therefore if we choose β sufficiently small such that $2(s-\alpha)/ns\beta > 1/\varepsilon$, then

$$\begin{aligned}
-M_6 &\geqslant \frac{1}{\beta}\min\left\{\frac{1-\varepsilon}{n}\left(\frac{s-\alpha}{s\beta}\right)^2 y^2 - \frac{n}{4(1-\varepsilon)}\left(\frac{s\beta}{s-\alpha}\right)^2 A^2 \right.\\
&\qquad \left. - 2\beta(s-1) M_4 \cdot v(2R)\, y^{1/2}\right\} \\
&= -\frac{n}{4(1-\varepsilon)}\left(\frac{s}{s-\alpha}\right)^2 A^2 \\
&\qquad - C_4^2\left(\frac{ns^2 M_4^4 (v(2R))^4 (s-1)^4}{(1-\varepsilon)(s-\alpha)^2}\right)^{1/3}.
\end{aligned} \tag{2.22}$$

By setting $\theta(2R)=k_2$, $M_3=k_3$, $M_5=k_4$, $\nu(2R)=k_5$, and $M_4=\max\{u^{\alpha-1}(x,t)\mid(x,t)\in B_P(2R)\times[0,\infty)\}$, and letting $R\to\infty$, $\varepsilon\to 0$, we note that (2.2) yields (2.21).

If $\alpha=1$, by (2.22), we have

$$M_6\leqslant\frac{n}{4(1-\varepsilon)}\left(\frac{s}{s-1}\right)^2 4K^2(2R)+C_4^2\left(\frac{ns^2(\nu(2R))^4(s-1)^2}{1-\varepsilon}\right)^{1/3}.$$

We set $s-1=\delta/R^2$, $K(2R)=0$, and $\theta(2R)=k_2$, and let $R\to\infty$, $\varepsilon\to 0$, $\delta\to 0$; then we can obtain (2.20) from (2.2).

Remark. The reuslt (II) is the main result in [5].

THEOREM 2.3. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor $R_{ij}\geqslant -k_1 g_{ij}$ $(k_1\geqslant 0)$. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,\infty)$ which is C^2 in the x-variable and C^1 in the t-variable. Assume that $0<\alpha<n/(n-1)$ $(n\geqslant 2)$, and $(\Delta+\partial/\partial t)\,h(x,t)\geqslant 0$.*

If $u(x,t)$ is a positive solution of (0.1), *then*

$$\frac{|\nabla u|^2}{u^2}+\frac{1}{\alpha}h\cdot u^{\alpha-1}-\frac{1}{\alpha}\frac{u_t}{u}\leqslant\frac{2n}{\alpha^2}\frac{1}{t}+\frac{2n}{\alpha(2-\alpha)}k_1. \tag{2.23}$$

Using Lemma 1.2 and an argument similar to Theorem 2.2, we can prove the theorem.

We can utilize the gradient estimates (2.23) to obtain the Harnack inequality for positive solutions of (0.1) by an argument similar to Theorem 2.2 in [5].

THEOREM 2.4. *Suppose M, $h(x,t)$, and $\alpha\in R^+$ satisfy the hypotheses of Theorem* 2.3. *If $u(x,t)$ is a positive solution of* (0.1), *then*

$$u(x_1,t_1)\leqslant u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{2n/\alpha}\exp\left(\frac{r^2(x_1,x_2)}{4\alpha(t_2-t_1)}+\frac{2nk}{(2-\alpha)}(t_2-t_1)\right), \tag{2.24}$$

where x_1, $x_2\in M$, $0<t_1<t_2<\infty$, and $r(x_1,x_2)$ is the geodesic distance between x_1 and x_2.

Proof. Since $h(x,t)\geqslant 0$, using Theorem 2.3, we know that if $f=\log u$, then

$$|\nabla f|^2-\frac{1}{\alpha}f_t\leqslant\frac{2n}{\alpha^2}\frac{1}{t}+\frac{2n}{\alpha(2-\alpha)}k_1 \tag{2.25}$$

for all $(x,t)\in M\times[0,\infty)$.

For any two points (x_1, t_1) and (x_2, t_2) in $M \times [0, \infty)$ with $t_1 < t_2$, we let r: $[0, 1] \to M$ be the shortest geodesic joining x_1 and x_2 with $r(0) = x_2$ and $r(1) = x_1$.

Define the curve η: $[0, 1] \to M \times [0, \infty)$ by $\eta(s) = (r(s), (1-s)t_2 + st_1)$. Clearly, $\eta(0) = (x_2, t_2)$, $\eta(1) = (x_1, t_1)$:

$$f(x_1, t_1) - f(x_2, t_2) = \int_0^1 \frac{df(\eta(s))}{ds}\, ds$$

$$\leqslant \int_0^1 (\rho\, |\nabla f| - (t_2 - t_1) f_t)\, ds, \tag{2.26}$$

where $\rho = r(x_1, x_2)$.

Applying inequality (2.25), $-f_t \leqslant -\alpha |\nabla f|^2 + (2n/\alpha)(1/t) + (2n/(2-\alpha))k_1$. So,

$$f(x_1, t_1) - f(x_2, t_2) \leqslant \int_0^1 \left(\rho\, |\nabla f| - \alpha(t_2 - t_1)\, |\nabla f|^2 \right.$$
$$\left. + \frac{2n}{\alpha}(t_2 - t_1)\frac{1}{t} + \frac{2n}{2-\alpha} k_1 (t_2 - t_1) \right) ds,$$

where $t = (1-s)t_2 + st_1$.

However, as a function of $|\nabla f|$, the quadratic

$$\rho\, |\nabla f| - \alpha(t_2 - t_1)\, |\nabla f|^2 + \frac{2n}{\alpha}(t_2 - t_1)\frac{1}{t} + \frac{2n}{2-\alpha} k_1 (t_2 - t_1)$$
$$\leqslant \frac{\rho^2}{4\alpha(t_2 - t_1)} + \frac{2n}{\alpha}(t_2 - t_1)\frac{1}{t} + \frac{2n}{2-\alpha} k_1 (t_2 - t_1).$$

So,

$$f(x_1, t_1) - f(x_2, t_2) \leqslant \frac{\rho^2}{4\alpha(t_2 - t_1)} + \frac{2n}{2-\alpha} k_1 (t_2 - t_1) + \frac{2n}{\alpha} \log\left(\frac{t_2}{t_1}\right),$$

which proves (2.24).

3. Global Behavior of Positive Solutions of Elliptic Equations

In this section, we first derive global gradient estimates for positive solutions $u(x)$ on M of Eq. (0.2) and then utilize the gradient estimates to obtain the Harnack inequality.

Finally, we prove a theorem of Liouville type.

THEOREM 3.1. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. Suppose that $h(x) \in C^2(M)$, $b(x) \in \mathfrak{X}(M)$, $\alpha \in R^+$, and R_{ij} satisfy the following conditions:*

(1) $\forall x \in M$, $\Delta h(x) + \nabla h(x) \cdot b(x) \geqslant 0$,

(2) $|b| \leqslant k_1$, $R_{ij} \geqslant -k_0 g_{ij}$, $b_{ij} \leqslant k_2 g_{ij}$ $(k_0, k_1, k_2 \geqslant 0)$,

(3) $1 < \alpha < n/(n-2)$ $(n \geqslant 4)$.

If $u(x)$ is a positive solution of (0.2), *then*

$$\frac{|\nabla u|^2}{u^2} + \frac{1}{\alpha} h u^{\alpha-1} \leqslant \frac{1}{\alpha(\alpha-1)} \frac{\sqrt{n(k_0+k_2) + \left(1+\frac{1}{\varepsilon'}\right) k_1}}{\left(\frac{2(1-\varepsilon')}{n} - \frac{\alpha-1}{\alpha}\right)^{1/2}}, \tag{3.1}$$

where $0 < \varepsilon' < 1 - (n/2)((\alpha-1)/\alpha)$.

Proof. Let P be a fixed point in M; $r(x)$ denotes the geodesic distance between P and x.

We define the function $F(x) = (R^2 - r^2(x))^2 \psi(x)$, where ψ is defined by (1.24) with $\nu = \beta^2/\alpha$.

Using the argument of Calabi [1] (also see [2]), we may assume that the function $F(x)$ with support in $B_P(R)$ is smooth.

Let x_1 be the point where F achieves its maximum in $B_P(R)$. We may assume $F(x_1) > 0$. Clearly, we have

$$\nabla F(x_1) = 0 \tag{3.2}$$

$$\Delta F(x_1) \leqslant 0. \tag{3.3}$$

Computing directly, (3.2) and (3.3) yield

$$\frac{\nabla \psi}{\psi} = \frac{4r \nabla r}{R^2 - r^2} \tag{3.4}$$

$$\frac{\Delta \psi}{\psi} - 8 \frac{r \nabla r}{R^2 - r^2} \frac{\nabla \psi}{\psi} - \frac{4r \Delta r}{R^2 - r^2} + \frac{8r^2}{(R^2 - r^2)^2} - \frac{4}{(R^2 - r^2)^2} \leqslant 0. \tag{3.5}$$

Using the Laplacian comparison theorem, we have

$$r \cdot \Delta r \leqslant (n-1)(1 + \sqrt{k_0}\, r). \tag{3.6}$$

By setting $\varepsilon'' = \frac{1}{2}$, $\varepsilon > 0$ sufficiently small such that $(2(1-\varepsilon)/n)(1-\varepsilon') > (\alpha-1)/\alpha$, and using Lemma 1.3, we note that (3.5) yields

$$\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon')-\frac{\alpha-1}{\alpha}\right]\frac{\alpha^2}{\beta^2}\psi+\left[\frac{1-\varepsilon}{n}\left(\frac{\alpha-\beta-1}{\beta}\right)^2-2\frac{1-\varepsilon}{\varepsilon}\right]\frac{|\nabla W|^4}{W^4}\frac{1}{\psi}$$
$$+\left[\frac{4(1-\varepsilon)}{n}\frac{\alpha(\beta+1-\alpha)}{\beta^2}+\frac{\alpha(\alpha-1)}{\beta^2}\right]\frac{|\nabla W|^2}{W^2}+\frac{2}{\beta}\frac{\nabla\psi}{\psi}\frac{\nabla W}{W}$$
$$-\frac{|\nabla\psi|}{\psi}k_1-2(k_0+k_2)\frac{|\nabla W|^2}{W^2}\frac{1}{\psi}-\frac{2(1-\varepsilon)}{n}\left(\frac{1}{\varepsilon'}+1\right)k_1\frac{|\nabla W|^2}{W^2}\frac{1}{\psi}$$
$$-8\frac{r\nabla r}{(R^2-r^2)}\frac{\nabla\psi}{\psi}-\frac{4r\Delta r}{R^2-r^2}+\frac{8r^2}{(R^2-r^2)^2}-\frac{4}{(R^2-r^2)^2}\leqslant 0. \tag{3.7}$$

Obviously,

$$2\left(k_0+k_2+\frac{1-\varepsilon}{n}\left(1+\frac{1}{\varepsilon'}\right)k_1\right)\frac{|\nabla W|^2}{W^2}\frac{1}{\psi}$$
$$\leqslant\frac{1-\varepsilon}{n}\delta\frac{(1-\alpha)^2}{\beta^2}\frac{|\nabla W|^4}{W^4}\frac{1}{\psi}+\frac{k_0+k_2+\frac{1-\varepsilon}{n}\left(1+\frac{1}{\varepsilon'}\right)k_1}{\frac{1-\varepsilon}{n}\delta(1-\alpha)^2}\beta^2\frac{1}{\psi}, \tag{3.8}$$

where $0<\delta<1$, and

$$\frac{2}{\beta}\frac{\nabla\psi}{\psi}\frac{\nabla W}{W}\leqslant\frac{4(1-\varepsilon)}{n}\frac{\alpha}{\beta}\frac{|\nabla W|^2}{W^2}+\frac{n}{4(1-\varepsilon)}\frac{1}{\alpha\beta}\frac{|\nabla\psi|^2}{\psi^2}. \tag{3.9}$$

Substituting (3.4), (3.6), (3.8), (3.9) into (3.7), and letting β be sufficiently small, we have

$$\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon')-\frac{\alpha-1}{\alpha}\right]\frac{\alpha^2}{\beta^2}\psi-\frac{n}{4(1-\varepsilon)}\frac{1}{\alpha\beta}\frac{|\nabla\psi|^2}{\psi^2}-\frac{|\nabla\psi|}{\psi}k_1$$
$$-\left[\frac{n}{\delta(1-\varepsilon)}\frac{k_0+k_2}{(1-\alpha)^2}+\frac{1}{\delta(1-\alpha)^2}\left(1+\frac{1}{\varepsilon'}\right)k_1\right]\beta^2\frac{1}{\psi}-24\frac{r^2}{(R^2-r^2)^2}$$
$$-\frac{4(n-1)(1+\sqrt{k_0}\,r)}{R^2-r^2}-\frac{4}{R^2-r^2}\leqslant 0. \tag{3.10}$$

Multiplying through by $(R^2-r^2)^4\psi$, (3.10) takes the form

$$\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon')-\frac{\alpha-1}{\alpha}\right]\frac{\alpha^2}{\beta^2}F^2-\left(\frac{4n}{1-\varepsilon}\frac{1}{\alpha\beta}+24\right)r^2F-4n(R^2-r^2)F$$
$$-[4k_1r+4(n-1)\sqrt{k_0}\cdot r]\cdot(R^2-r^2)F-\left[\frac{n}{\delta(1-\varepsilon)}\frac{k_0+k_2}{(1-\alpha)^2}\right.$$
$$\left.+\frac{1}{\delta(1-\alpha)^2}\left(1+\frac{1}{\varepsilon'}\right)k_1\right]\cdot\beta^2(R^2-r^2)^4\leqslant 0. \tag{3.11}$$

Equation (3.11) implies

$$F \leqslant \frac{\beta^2}{\alpha^2} \frac{1}{\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon') - \frac{\alpha-1}{\alpha}\right]}$$
$$\times \left[\left(\frac{4n}{1-\varepsilon}\frac{1}{\alpha\beta} + 24 + 4n\right) R^2 + (4k_1 + 4(n-1)\sqrt{k_0}) R^3\right]$$
$$+ \frac{\beta^2}{\alpha} \frac{R^4}{\left(\frac{2(1-\varepsilon)}{n}(1-\varepsilon') - \frac{\alpha-1}{\alpha}\right)^{1/2}}$$
$$\times \sqrt{\frac{n}{\delta(1-\varepsilon)}\frac{k_0+k_2}{(1-\alpha)^2} + \frac{1}{\delta(1-\alpha)^2}\left(1+\frac{1}{\varepsilon'}\right) k_1}.$$

Therefore,

$$\sup_{x \in B_P(\delta' R)} \psi(x) \leqslant \frac{\beta^2}{\alpha^2} \frac{1}{\left[\frac{2(1-\varepsilon)}{n}(1-\varepsilon') - \frac{\alpha-1}{\alpha}\right]} \left[\left(\frac{4n}{1-\varepsilon}\frac{1}{\alpha\beta} + 24 + 4n\right) R^2\right.$$
$$\left. + (4k_1 + 4(n-1)\sqrt{k_0}) R^3\right] \frac{1}{(1-(\delta')^2)^2 R^4} + \frac{\beta^2}{\alpha}\frac{1}{(1-(\delta')^2)^2}$$
$$\times \frac{1}{\left(\frac{2(1-\varepsilon)}{n}(1-\varepsilon') - \frac{\alpha-1}{\alpha}\right)^{1/2}} \left[\frac{n}{\delta(1-\varepsilon)}\frac{k_0+k_2}{(1-\alpha)^2}\right.$$
$$\left. + \frac{1}{\delta(1-\alpha)^2}\left(1+\frac{1}{\varepsilon'}\right) k_1\right]^{1/2}. \tag{3.12}$$

Since $\psi = \beta^2(|\nabla u|^2/u^2) + (\beta^2/\alpha)\, hu^{\alpha-1}$, letting $R \to \infty$, $\varepsilon \to 0$, $\delta' \to 0$, $\delta \to 1$, we have

$$\frac{|\nabla u|^2}{u^2} + \frac{1}{\alpha} hu^{\alpha-1}$$
$$\leqslant \frac{1}{\alpha(\alpha-1)} \frac{1}{\left(\frac{2(1-\varepsilon')}{n} - \frac{\alpha-1}{\alpha}\right)^{1/2}} \sqrt{n(k_0+k_2) + \left(1+\frac{1}{\varepsilon'}\right) k_1}.$$

If $h(x)$ is nonnegative, we can easily obtain the Harnack inequality for a positive solution of (0.2) usinig the gradient estimate (3.1).

COROLLARY 3.1. *Suppose that M, α, b, h satisfy the hypotheses of Theorem* 3.1, *and in addition we assume that $h(x) \geqslant 0$. If $u(x)$ is a positive solution of* (0.2), *then*

$$\operatorname*{Sup}_{B_P(R)} u(x) \leqslant \exp(C_{n,\alpha,k_0,k_1,k_2,\varepsilon'} \cdot R) \cdot \inf_{B_P(R)} u(x).$$

Finally, we prove the following theorem of Liouville type.

THEOREM 3.2. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. Suppose that $h(x) \in C^2(M)$, $b(x) \in \mathfrak{X}(M)$, $\alpha \in R^+$, and R_{ij} satisfy the following conditions*:

(1) $\forall x \in M$, $\Delta h(x) + \nabla h(x) \cdot b(x) \geqslant 0$;

(2) $1 < \alpha < n/(n-2)$ $(n \geqslant 4)$;

(3) *the tensor field*

$$R_{ij} - \nabla_i b_j - \frac{1}{n}\left(1 + \frac{1}{\varepsilon'}\right) |b| \, g_{ij} \tag{3.13}$$

is positive definite, where ε' is a positive constant such that $2(1-\varepsilon')/n > (\alpha - 1)/\alpha$;

(4) $|b| = o(r)$ *as* $r \to \infty$, $R_{ij} \geqslant -f(r)\, g_{ij}$ *with* $f(r) \geqslant 0$, *and* $f(r) = o(r^2)$ *as* $r \to \infty$, *where $r(x)$ is the geodesic distance from x to some fixed point P in M*;

(5) *there exists a point $x_0 \in M$ such that $h(x_0) \geqslant 0$.*

Then Eq. (0.2) *does not have a positive solution on M.*

Proof. By an argument similar to Theorem 3.1, we know that if $u(x)$ is a positive solution of (0.2) on M then

$$\frac{|\nabla u|^2}{u^2} + \frac{1}{\alpha} h u^{\alpha - 1} \leqslant 0. \tag{3.14}$$

This implies $h(x) \leqslant 0$ on M.

Since $h(x_0) \geqslant 0$ and $h(x)$ satisfies the condition (1), $h(x)$ can not have maximum at x_0; $h(x)$ must achieve some positive value on M.

This completes the proof of the theorem.

Remark. We suspect that Theorems 3.1 and 3.2 also hold when $n = 3$.

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